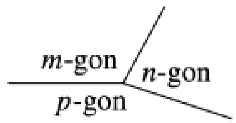
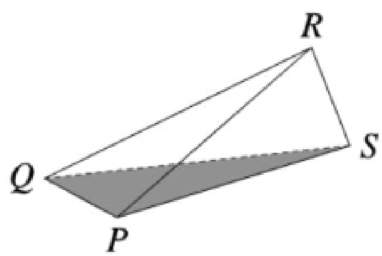
Problems from Crux MathemAttic, due February 15, 2024

MA246. The length of a rectangular piece of paper is three times its width. The paper is folded so that one vertex lies on top of the opposite vertex, thus forming a pentagonal shape. What is the area of the pentagon as a fraction of the area of the original rectangle?

MA247. A regular m-gon, a regular n-gon and a regular p-gon share a vertex and pairwise share edges, as shown in the diagram. What is the largest possible value of p?

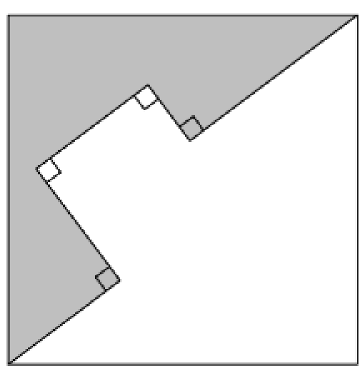


MA248. A drinks carton is formed by arranging four congruent triangles as shown. We have $QP = RS = 4$ and $PR = PS = QR = QS = 10$. What is the volume of the carton?



MA249. Peter has 25 cards, each printed with a different integer from 1 to 25. He wishes to place N cards in a single row so that the numbers on every adjacent pair of cards have a prime factor in common. What is the largest value of N for which this is possible?

MA250. Five line segments of length 2, 2, 2, 1 and 3 connect two corners of a square as shown in the diagram. What is the shaded area?



Problems from Crux Olympiad Corner, due February 15, 2024

OC656. Let $\mathbb{P}$ be the set of all prime numbers. Find all functions $f : \mathbb{P} \to \mathbb{P}$ such that:

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all $p, q \in \mathbb{P}$.

OC657. Determine all real numbers x such that

$$x - \sqrt{x}\log_{1/\sqrt{2}} x \geq \sqrt{x} - x\log_{1/\sqrt{2}} x.$$

OC658. A quadrilateral $ABCD$ is inscribed in a circle k where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of diagonals AC and BD, and the perpendicular from M to AB intersects the segment AB at a point E. If EM bisects the angle CED, prove that AB is a diameter of k.

OC659. Find all sequences of integers $a_0, a_1, a_2, \ldots$ such that for any integers $k, l \geq 0$, we have

$$a_k - a_l \mid k^2 - l^2.$$

That is for any integers $k, l \geq 0$ there exists an integer z such that

$$(a_k - a_l)z = k^2 - l^2.$$

OC660. Let ABC be a triangle, and M the midpoint of the side BC. Let E and F be points on the sides AC and AB, respectively, so that $ME = MF$. Let D be the second intersection of the circumcircle of MEF and the side BC. Consider the lines ℓ_D, ℓ_E and ℓ_F through D, E and F, respectively, such that $\ell_D \perp BC$, $\ell_E \perp CA$, and $\ell_F \perp AB$. Show that ℓ_D, ℓ_E and ℓ_F are concurrent.

Problems from Crux Mathematicorum, due February 15, 2024

4891. *Proposed by Mihaela Berindeanu.*

Let x, y, z be nonegative real numbers such that $x + y + z > 0$. Show that

$$\sqrt{4x+y+z}+\sqrt{x+4y+z}+\sqrt{x+y+4z} \geq 18\frac{\sqrt{xy+xz+yz}}{\sqrt{x+y}+\sqrt{x+z}+\sqrt{y+z}}.$$

(*Hint.* You may consider the points $A = (\sqrt{x}, 0, 0), B = (0, \sqrt{y}, 0), C = (0, 0, \sqrt{z})$ in an Euclidean system of coordinates.)

4892. *Proposed by Dong Luu.*

Consider an acute triangle $ABC, (AB < AC)$. Let (I), (O) be the inscribed and circumscribed circles, respectively. Let D, E, F be the points of contact of (I) and BC, CA, AB and let H be the midpoint of arc BAC. Suppose HD cuts (O) a second time at K and suppose AK cuts EF at J. Finally, let M, N be midpoints of BC and AJ, respectively. Prove that the four points M, N, O, K lie on the same circle.

4893. *Proposed by Albert Natian.*

Find all continuous real functions f on $[-1, 1]$ that satisfy the integral equation

$$x^2 + \int_1^{\frac{1}{x}} f\left(x^2 t\right) dt = 1.$$

4894. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

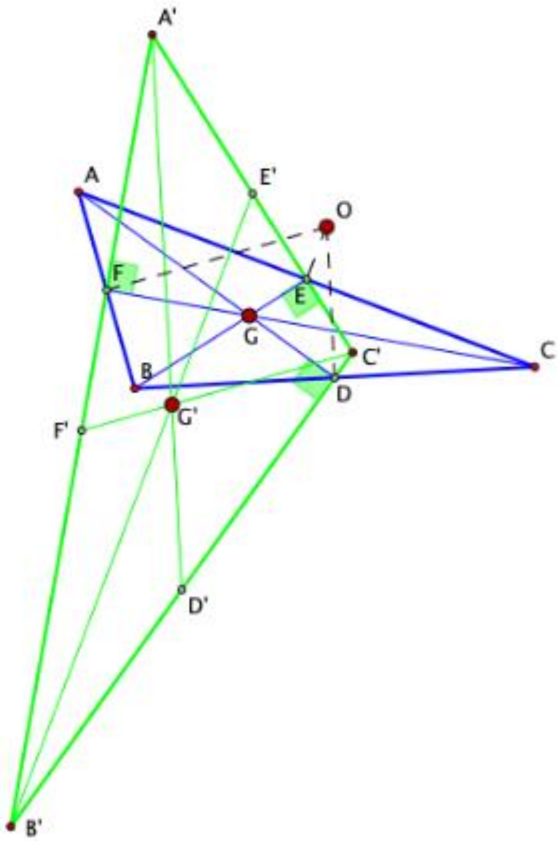
Calculate

$$\sum_{n=1}^{\infty} \frac{H_{n-1}H_{n+1}}{n(n+1)},$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number and $H_0 = 0$.

4895. *Proposed by Shawn Godin and J. Chris Fisher.*

For any triangle ABC with centroid G and circumcenter O, denote by G' the centroid of the triangle $A'B'C'$ formed by the lines through the midpoints D, E, and F of the sides that are perpendicular to the corresponding medians (namely, AD, BE, CF). In the accompanying diagram, for example, the side labeled $B'C'$ of the companion triangle $A'B'C'$ is perpendicular to the median AD to the side BC of the given triangle. Prove that G is the midpoint of the segment $G'O$.



4896. *Proposed by Ivan Hadinata.*

Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$ the following equation holds:

$$f(f(x) + yf(z) - 1) + f(z + 1) = zf(y) + f(x + z).$$

4897. *Proposed by Michel Bataille.*

Let the sequence $(x_n)_{n\geq 0}$ be defined by $x_0 = 2, x_1 = 7$ and $x_{n+1} = 4x_n - x_{n-1}$ for all $n \geq 1$. Prove that for all $n \geq 0$,

$$\lfloor\sqrt{8x_{2n+1} + 6}\rfloor - \lfloor\sqrt{2x_{2n+1} + 1}\rfloor = 2x_n.$$

4898. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with $\angle B = 2\angle C$. Consider an interior point D on the side BC such that $\angle C = 2\angle CAD$. If b and c are the lengths of sides AC and AB respectively, prove that

$$AD = \frac{b}{b+c}\sqrt{bc + 2c^2}.$$

4899. *Proposed by Aravind Mahadevan.*

Let ABC be a right-angle triangle with $\angle B = 90^\circ$. Suppose AE and CD are angle bisectors of angles BAC and ACB, respectively. If $AE = 9$ and $CD = 8\sqrt{2}$, find the length of AC.

4900. *Proposed by Daniel Sitaru.*

For a positive integer m, let H_m denote the m-th harmonic number, that is, $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$. For m, n, p, q positive integers, prove that

$$H_m + H_n + H_p + H_q \leq 3 + H_{mnpq}.$$

Problems from the American Mathematical Monthly, due February 29, 2024

12412. *Proposed by Richard Stanley, University of Miami, Coral Gables, FL.* For $n \geq 1$, let $f(n) = \sum_d d^{n/d}\ (n/d)!$, where the sum is over all positive squarefree divisors of n. Prove that $f(n)$ is divisible by n^2.

12413. *Proposed by Seewoo Lee, Berkeley, CA.* For a positive real number r, let $I_r = \int_0^{\pi/2} \sin^r \theta \, d\theta$. Prove

$$\frac{1}{(r+1)^2} + I_{r+1}^2 < \left(\frac{r+3}{r+2}\right)^2 I_r^2$$

for all $r \geq 1$.

12414. *Proposed by Florentin Visescu, Bucharest, Romania.* Let P be a point inside triangle ABC. Prove

$$\cos\frac{\angle ABC}{2}\sin\frac{\angle APC}{2} + \cos\frac{\angle BCA}{2}\sin\frac{\angle BPA}{2} + \cos\frac{\angle CAB}{2}\sin\frac{\angle CPB}{2} \leq \frac{9}{4},$$

with equality if and only if ABC is equilateral and P is its center.

12415. *Proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.* For a nonnegative integer n, evaluate

$$\sum_{j=0}^{2n} \sum_{k=\lfloor j/2 \rfloor}^{j} \binom{2n+2}{2k+1}\binom{n+1}{2k-j}.$$

12416. *Proposed by Mihai Opincariu, Brad, Romania, and Vasile Pop, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let A and B be n-by-n complex matrices such that $\operatorname{rank}(AB) = \operatorname{rank}(BA)$. Prove that $AB^2A = AB$ if and only if $BA^2B = BA$.

12417. *Proposed by Mohsen Maesumi, Lamar University, Beaumont, TX.* Consider the sphere S given by $x^2 + y^2 + (z-1)^2 = 1$, with north pole N at $(0, 0, 2)$. The stereographic projection of a point P at $(x, y, 0)$ is the point, different from N, that is on the intersection of NP with S. Consider the region H in the xy-plane given by $0 \leq xy \leq c^2$, where $c > 0$. What is the area of the stereographic projection of H to S?

12418. *Proposed by Vladimir Lucic, Imperial College, London, UK.* Let Φ be the cumulative distribution function of a standard normal random variable, defined by $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2}\, dt$.

(**a**) For positive real numbers $\sigma_1, \ldots, \sigma_n$ and $w_1, \ldots, w_n$ with $\sum_{i=1}^n w_i = 1$, determine

$$\lim_{x\to\infty} \frac{1}{x}\Phi^{-1}\left(\sum_{i=1}^{n} w_i \Phi\left(\frac{x}{\sigma_i}\right)\right).$$

(**b**) Let L be the limit in (**a**). Determine

$$\lim_{x\to\infty} x^2\left(\frac{1}{x}\Phi^{-1}\left(\sum_{i=1}^{n} w_i \Phi\left(\frac{x}{\sigma_i}\right)\right) - L\right).$$

Problems from Crux MathemAttic, due March 15, 2024

MA251. Bob is practicing addition in base 2. Each time he adds two numbers in base 2, he counts the number of carries. For example, when summing the numbers 1001 and 1011 in base 2, there are three carries (shown on the top row).

1		1	1	
0	1	0	0	1
0	1	0	1	1
1	0	1	0	0

Suppose that Bob starts with the number 0 and adds 111 (i.e. 7 in base 2) to it one hundred times to obtain the number 1010111100 (i.e. 700 in base 2). How many carries occur (in total) in these one hundred calculations?

MA252. An Indian raga has two kinds of notes: a short note, which lasts for 1 beat, and a long note, which lasts for 2 beats. For example, there are 3 ragas which are 3 beats long: 3 short notes, a short note followed by a long note, and a long note followed by a short note. How many Indian ragas are 11 beats long? Justify your answer.

MA253. Let $n \geq 2$ be an integer. There are n houses in a town. All distances between pairs of houses are different. Every house sends a visitor to the house closest to it. Find all possible values of n (with full justification) for which we can design a town with n houses where every house is visited.

MA254. A sequence $a_1, a_2, \ldots$ satisfies $a_1 = \frac{5}{2}$ and $a_{n+1} = a_n^2 - 2$ for all $n \geq 1$. Let M be the integer closest to a_{2023}. Find the last digit of M.

MA255. A $3 \times 3 \times 3$ cube of cheese is sliced into twenty-seven $1 \times 1 \times 1$ blocks. A mouse starts anywhere on the outside and eats one of the $1 \times 1 \times 1$ cubes. He then moves to an adjacent cube (in any direction), eats that cube, and continues until he has eaten all 27 cubes. (Two cubes are considered adjacent if they share a face.) Prove that no matter what strategy the mouse uses, he cannot eat the middle cube last. (Note: One should neglect gravity: intermediate configurations dont collapse.)

Problems from Crux Olympiad Corner, due March 15, 2024

OC661. Point N is the midpoint of side AD of a convex quadrilateral $ABCD$, and point M on side AB is such that $CM \perp BD$. Prove that if $BM > MA$, then $2BC + AD > 2CN$.

OC662. Let $a_1, \ldots, a_k$ be distinct positive integers such that the difference between the largest and smallest of them is less than 1000. What is the largest k for which it is possible that all quadratic equations $a_i x^2 + 2a_{i+1}x + a_{i+2} = 0$, where $1 \le i \le k-2$, have no real roots?

OC663. There are 100 cities in the Far Far Away Kingdom, and every two cities are connected by no more than one road. One day the king ordered the introduction of one-way traffic on every road, and at the same time every road was painted white or black. The Minister of Transport proudly announced that after carrying out the order, one can get from any city to any other along roads alternating their colors, and so that the first road along the way will be white. What is the smallest number of roads there could be in this country? When getting from city to city, you can pass through intermediate cities any number of times.

OC664. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

OC665. Let A, B and C be $n \times n$ matrices with complex entries satisfying

$$A^2 = B^2 = C^2 \qquad \text{and} \qquad B^3 = ABC + 2I.$$

Prove that $A^6 = I$.

Problems from Crux Mathematicorum, due March 15, 2024

4901. *Proposed by Michel Bataille.*

Let ABC be a triangle and I its incenter. Let M, N on the line BI and P, Q on the line CI be such that AM, CN (resp. AP, BQ) are perpendicular to BI (resp. CI). Prove that M, N, P, Q are concyclic and that MP is parallel to BC.

4902. *Proposed by Titu Zvonaru.*

Let O be the circumcenter of triangle ABC. Let AN be the altitude from A. Lines BO and CO intersect the lines AC and AB at E and F, respectively. Prove that if $\angle BAC = 45°$, then the lines NO and EF are perpendicular.

4903. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty} \left[\left(\frac{1}{2n-1} - \frac{1}{2n+1} + \frac{1}{2n+3} - \cdots \right) - \frac{1}{4n} \right].$$

4904. *Proposed by Ivan Hadinata.*

Find all pairs (x, y) of prime numbers x and y such that $x \geq y$, $x+y$ is prime and $x^x + y^y$ is divisible by $x+y$.

4905. *Proposed by Aravind Mahadevan.*

In a right-angled triangle, the acute angles x and y satisfy the following equation:

$$\tan x + \tan y + \tan^2 x + \tan^2 y + \tan^3 x + \tan^3 y = 70.$$

Find x and y.

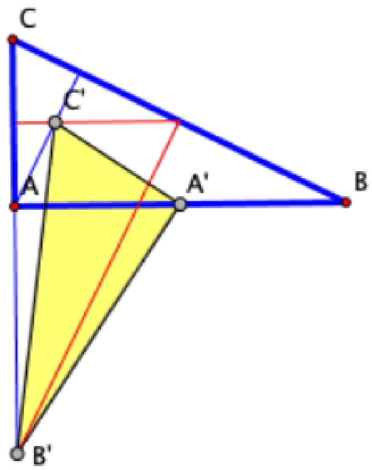
4906. *Proposed by Cristinel Mortici.*

Find positive integers m and n such that $m^3 + n + 12$ is a perfect cube and $n^2 + m + 13$ is a perfect square.

4907. *Proposed by J. Chris Fisher.*

Given triangle ABC with a right angle at A, define A' to be the midpoint of the leg AB, B' to be the point where the perpendicular bisector of BC intersects the line AC, and C' to be the point where the perpendicular bisector of AC intersects the altitude from A to BC.

Prove that $\Delta A'B'C'$ is similar to ΔABC.



4908. *Proposed by Mihaela Berindeanu.*

In the square $ABCD$, the points X, Y and Z are respectively on the segments AB, AD and AX so that $XC = XY$ and $\angle ZYX = \angle XCB$. Show that $AY \cdot ZC^2 = 2 \cdot ZB \cdot BC \cdot YZ$.

4909. *Proposed by Michel Bataille.*

For each positive integer n, let $P_n(x) = (x-1)^{2n+1}(x^2-(2n+1)x-1)$. Show that the equation $P_n(x) = 1$ has a unique solution x_n in the interval $(0, \infty)$. Prove that $\lim\limits_{n\to\infty}(x_n - 2n) = 1$ and find $\lim\limits_{n\to\infty} n(x_n - 2n - 1)$.

4910. *Proposed by Paul Bracken.* Let m and n be non-negative integers and let

$$J_{m,n} = \int_0^\infty \left(\left(\frac{\sin t}{t}\right)^m - \left(\frac{\sin t}{t}\right)^n \right) \frac{dt}{t^2}.$$

Prove that the $J_{m,n}$ are rational multiples of π.

Problems from Mathematics Magazine, due March 1, 2024

2176. *Proposed by Elton Bojaxhiu, Eppstein am Taunus, Germany and Enkel Hysnelaj, Sydney, Australia.*

Show that

$$\int_0^1 \frac{\log\left(x^2+x+1\right)}{x^2+1}\,dx = \frac{1}{6}\pi \log\left(\sqrt{3}+2\right) - \frac{C}{3},$$

where $C = 1/1^2 - 1/3^2 + 1/5^2 - 1/7^2 + \cdots$ is the Catalan constant.

2177. *Proposed by Hidefumi Katsuura, San Jose State University, San Jose, CA.*

Prove that in any triangle with side lengths a, b, c, inradius r, and circumradius R, we have

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{r}{R} \leq 2.$$

Note: Problem 2077 in the October 2019 issue of *Mathematics Magazine* asks the reader to show that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} + \frac{r}{R} > \frac{5}{3}.$$

2178. *Proposed by the Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*

(a) Find the maximum volume of a right circular cone inscribed in a unit sphere.
(b) Find the maximum total surface area of a right circular cone inscribed in a unit sphere.
(c) In four dimensions, find the maximum hypervolume of a hypercone inscribed in a unit hypersphere.
(d) In four dimensions, find the maximum total surface volume of a hypercone inscribed in a unit hypersphere.

For parts (c) and (d), see the Wikipedia article "Hypercone".

2179. *Proposed by Jacob Siehler, Gustavus Adolphus College, Saint Peter, MN.*

A puzzle consists of a 6×4 board with an initial configuration of markers on it. The goal is to add markers so there are exactly two markers in every row and exactly three markers in every column. What is the minimum number of markers in an initial configuration having a unique solution?

2180. *Proposed by Philippe Fondanaiche, Paris, France.*

(a) Find the smallest integer k such that in the sequence of k distinct positive integers, $a_1, \ldots, a_k$, each of the integers 1, 2, 3, and 4 (not necessarily in this order) appears exactly once and a_1a_2 is a perfect square, $a_1a_2a_3$ is a perfect cube, ..., $a_1a_2 \ldots a_k$ is a perfect kth power.
(b) Does there exist an infinite sequence a_i, $i = 1, 2, 3, \ldots$, of positive integers such that every positive integer appears exactly once in the sequence and

$$\prod_{i=1}^{n} a_i \text{ is a perfect } n\text{th power}$$

for all $n \geq 1$?

Problems from the College Mathematics Journal, due March 15, 2024

1251. *Proposed by Cezar Lupu, Yanqi Lake BIMSA Tsinghua University, Beijing, China.*

Let n be a positive integer and A be an $n \times n$ matrix with integer entries. Further, suppose that $\det(A - I_n) \neq 0$, where I_n is the $n \times n$ identity matrix. Finally, assume that $A + A^2 + \cdots + A^p = pI_n$, where p is a prime number. Prove that $p - 1$ is a factor of n.

1252. *Proposed by Narendra Bhandari, Bajura District, Nepal.*

Prove that

$$\sum_{n=1}^{\infty} \binom{2n}{n} \binom{2n-2}{n-1} \frac{1}{2^{4n-1}(2n-1)} = \frac{1}{\pi}.$$

1253. *Proposed by Reza Farhadian, Razi University, Kermanshah, Iran, and Rafael Jakimczuk, Universidad Nacional de Luján, Buenos Aires, Argentina.*

Let $\alpha > 0$ be a real number and let k be a positive integer. Evaluate the following limit.

$$\lim_{n\to\infty} \left(\frac{1}{(k(n+1))^{k\alpha}} \cdot \frac{\sum_{i=1}^{n+1} (ki)^{k\alpha i}}{\sum_{i=1}^{n} (ki)^{k\alpha i}} \right)$$

1254. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Central University of Cluj-Napoca, Cluj-Napoca, Romania.*

Evaluate the following limit.

$$\lim_{n\to\infty} n^2 \int_0^1 \left(\frac{1}{n} + x^n \right)^n dx$$

1255. *Proposed by Albert Natian, Los Angeles Valley College, Valley Glen, CA.*

Let ν be a positive integer, and let $a \leq \nu$ be an integer. Let $X_0 = a$, and let $X_1, X_2, X_3, \ldots$ be a countably infinite sequence of independent random variables uniformly taking values in the set $\mathbb{N}_\nu := \{1, 2, \ldots, \nu\}$. Now let $M_a := \min\{n\colon X_n < X_{n-1}\}$ and $N_a := \min\{n\colon X_n \leq X_{n-1}\}$. Compute the following expected values.

1. $E(a, \nu) := E[M_a]$, and
2. $F(a, \nu) := E[N_a]$.

Problems from the American Mathematical Monthly, due March 31, 2024

12419. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* The terms of an alternating series are the consecutive (from left to right) base-ten digits of the consecutive positive integers starting with 1:

$$1-2+3-4+5-6+7-8+9-1+0-1+1-1+2-1+3-1+4-\cdots.$$

Prove that for any integer n, this series has infinitely many partial sums equal to n.

12420. *Proposed by Seon-Hong Kim, Sookmyung Women's University, Seoul, Korea, and Kenneth Stolarsky, University of Illinois, Urbana, IL.* For a polynomial p of positive degree, let p^* be p with its leading term deleted. Show that there are arbitrarily large integers N for which there is a polynomial p with integer coefficients such that p has exactly N zeros on the unit circle and p^* has at least $2N$ zeros on the unit circle.

12421. *Proposed by Ioan Caşu, West University of Timişoara, Timişoara, Romania.* Let S be a finite set of real numbers, and let T be the set of all n-by-n matrices having entries in S. Prove

$$\sum_{A\in T}\operatorname{trace}\left(A^2\right)=\sum_{A\in T}(\operatorname{trace}(A))^2.$$

12422. *Proposed by Mohammed Aassila, Strasbourg, France.* Let a, b, c be integers such that $a\neq 0$ and $an^2+bn+c\neq 0$ for all positive integers n.
(a) Prove that if there is a positive integer k such that $b^2-4ac=k^2a^2$, then

$$\sum_{n=1}^{\infty}\frac{1}{an^2+bn+c}$$

is rational.
(b)* Is the converse of **(a)** true?

12423. *Proposed by Michel Bataille, Rouen, France.* For which primes p does there exist a group containing an element a of order 11 and an element b of order p such that $ba = ab^2$?

12424. *Proposed by Isaac Browne, Irvine, CA, and Edward Hou, Carnegie Mellon University, Pittsburgh, PA.* Prove that there is a sequence $L_1, L_2, \ldots$ of congruent convex sets in the plane such that for every finite set S of positive integers, the intersection

$$\bigcap_{i \in S} L_i \cap \bigcup_{i \notin S} L_i^c$$

has nonempty interior.

12425. *Proposed by Mihai Ciucu, Indiana University, Bloomington, IN.* The regular tetrahedron can be projected orthogonally onto a plane so that the projection is an equilateral triangle.

(a) Is it possible to cut a cube into two pieces by a single plane cut and rearrange the pieces to form a polyhedron having an orthogonal projection onto a plane that is an equilateral triangle? (Here a polyhedron need not be convex, but must have connected interior.)

(b) Same question for the octahedron.

(c)* Same question for the dodecahedron.

(d)* Same question for the icosahedron.